PMM U.S.S.R., Vol. 46, No. 5, pp. 679-686, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 532.52:539.3

ON A DYNAMIC CONTACT PROBLEM FOR COMPOUND FOUNDATION*

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The two-dimensional problem (plane state of strain) for interaction of a moving solid (die) with a layer of an ideal incompressible liquid of finite depth is investigated. The pressure from the die on the hydraulic foundation is transmitted through a thin covering. The die moves along the boundary of the covering at a constant speed without friction. This type of problem arises, for instance, when studying processes produced in the dynamic effects of solids on the surface of an ice cover.

Using the methods of the operational calculus in a moving coordinate system, the problem is reduced to finding the pressure function under the die from an integral equation of the first kind with a difference kernel. The Fourier transform of the latter has singularities on the real axis that determine the shape of the surface of the covering outside the die. Different shapes of the die foundations are discussed, and the characteristic features of the solution of the resulting integral equation are studied. Conditions for complete adherence of the die and the foundation are clarified, and examples in which the die pulls apart from the surface of the covering are studied. A numerical analysis of the problem for different shapes of the foundations of the die is presented.

1. Suppose an elastic layer (G, ν) of small thickness h lies on the surface of a layer $(H \leqslant y \leqslant 0)$ of a heavy ideal incompressible liquid. We assume that a rigid die clamped to the foundation with a force P and eccentricity e of the application of the force relative to the center of the die moves, along the boundary of such compound foundation, without friction at a constant speed W. We will also assume that in the course of motion the die does not cause the covering to peel off the liquid. We suppose that in a moving coordinate system bound to the die $(x'O'y)$, its foundation is described by the function $y = f(x')$, and that the line of contact is determined by the inequality $-c \leqslant x' \leqslant b$.

The physico-mechanical properties of a thin covering will be described by the Kirchhoff–Love plate equations for the case of a lengthwise constant force σ:

$$Dv^{(4)} - h\sigma v'' = p^*(x, t) - q^*(x, t) - h\gamma^* v^{\cdot\cdot} \qquad (1.1)$$
$$D = Gh^3 [6(1-\nu)]^{-1}$$

with $\sigma > 0$ corresponding to tensile forces while $\sigma < 0$, compressive forces. In (1.1), v denotes the displacement of the points of the mid-plane of the covering along the y-axis, $p^*(x, t) = p(x')$ is the reaction pressure on the layer from the direction of the liquid, and $q^*(x, t) = q(x')$ is the contact pressure (which is nonzero only when $-c \leqslant x' \leqslant b$, $x' = x + Wt$ and γ^* is the density of the material of the covering).

To describe the mechanical properties of the liquid, we will use the approximation theory of waves of low amplitude /1/, which may be derived from the basic exact theory of irrotational flows of an ideal incompressible liquid by linearization of the conditions imposed on the free surface, under the condition that the particular wave motion differs little from undisturbed flow from a horizontal free surface. Thus, we have

$$\Delta\varphi = 0, \quad v_x = \frac{\partial\varphi}{\partial x'} + W, \quad v_y = \frac{\partial\varphi}{\partial y}, \quad p = -\rho\left(W\frac{\partial\varphi}{\partial x'} + gy\right) \qquad (1.2)$$

Here $\varphi(x', y)$ is the velocity potential; p, increment in pressure in the liquid; ρ, density of liquid; g, gravitational constant; and v_x and v_y, projections of the velocities of the particles of the liquid on the axes of the moving coordinate system.

It is know that

$$v = -[\delta + \alpha x' - f(x')] \quad (-c \leqslant x' \leqslant b) \qquad (1.3)$$

*Prikl.Matem.Mekhan.,Vol.46,No.5,pp.847-856,1982 by virtue of the contact condition between the die and the covering, where $\delta + \alpha x'$ is the rigid displacement of the die under the influence of the force P and moment $M = Pe$ applied to it.

Because the covering is relatively thin, we may remove the condition (1.1) from the mid-plane at the boundary of the liquid layer $y = 0$. In the moving coordinate system, we will have

$$Dv^{(4)} - Tv'' = p(x') - q(x'),\ T = h(\sigma - \gamma^* W) \tag{1.4}$$

Following /1/, the contact condition between the liquid and the surface of the covering is written in the form

$$\partial\varphi/\partial y = W\partial v/\partial x' \tag{1.5}$$

Formula (1.4) and the last relation in (1.2) may be represented in accordance with (1.5) (when $y = 0$) on the form

$$\frac{\partial^2}{\partial x\, \partial y}\left(D\frac{\partial^2\varphi}{\partial x^2} - T\varphi\right) = W[p(x) - q(x)] \tag{1.6}$$
$$\frac{\partial p}{\partial x} = -\rho\left(W\frac{\partial^2\varphi}{\partial x^2} + gW^{-1}\frac{\partial\varphi}{\partial y}\right)$$

Here and below, the prime will be omitted from the moving coordinate x'.

To produce a closed formulation of the boundary-value problem (1.2), (1.6), it is necessary to add a non-flow-through condition for the foundation of the layer of liquid:

$$v_y(x, -H) = 0 \tag{1.7}$$

By means of an integral Fourier transformation, we solve the differential equation (1.2) for $\varphi(x, y)$ under the boundary conditions (1.6), (1.7). We obtain the following expression for the displacement v at $y = 0$:

$$v(x) = -\frac{1}{2\pi}\int_{-c}^{b} q(\xi)\, d\xi \int_\Gamma \frac{e^{-iu(x-\xi)}\, du}{A_0u^4 + A_2u^2 - A_3u \operatorname{cth} Hu + A_4} \tag{1.8}$$
$$A_0 = D,\ A_2 = T,\ A_3 = \rho W^2,\ A_4 = \rho g$$

Note that the expression in the second integral in (1.8) may have four single real poles $\pm c_1$ and $\pm c_2$ symmetrically located relative to the origin. The path Γ is selected in accordance with the principle of maximal absorption and coincides everywhere with the real axis, deviating from it in passing around the two large (in absolute value) poles from below, and two other poles from above.

Now using condition (1.3), we obtain an integral equation for the unknown contact pressures $q(x)$ under the die. In dimensionless variables and with the notation

$$\xi = \xi'\frac{b+c}{2} + \frac{b-c}{2},\quad x = x'\frac{b+c}{2} + \frac{b-c}{2}$$
$$\lambda = \frac{2h}{b+c},\quad \beta = \frac{H}{h}$$
$$\delta' = \delta\frac{2}{b+c} + \frac{b-c}{b+c},\quad f\left(x'\frac{b+c}{2} + \frac{b-c}{2}\right)\frac{2}{b+c} = r(x')$$
$$q\left(\xi'\frac{b+c}{2} + \frac{b-c}{2}\right)h^3D^{-1} = g(\xi'),\quad A_i' = A_iD^{-1}h^i \quad (i = 2, 3, 4)$$

we obtain

$$\int_{-1}^{1} g(\xi) K\left(\frac{x-\xi}{\lambda}\right) d\xi = \pi[\delta + \alpha x - r(x)] \quad (|x| \leqslant 1) \tag{1.9}$$

$$K(z) = \frac{1}{2}\int_\Gamma L(u) e^{-iuz}\, du \tag{1.10}$$
$$L(u) = \frac{\operatorname{sh}\beta u}{(u^4 + A_2u^2 + A_4)\operatorname{sh}\beta u - A_3u \operatorname{ch}\beta u}$$

Here and bleow, the prime will be omitted from the dimensionless variables and the notation.

Note that we must add one more static condition to equation (1.9):

$$N_0 = 2Ph^3D^{-1}(b+c)^{-1} = \int_{-1}^{1} g(x)\,dx \tag{1.11}$$

$$N_1 = 4Peh^3D^{-1}(b+c)^{-2} = \int_{-1}^{1} xg(x)\,dx + \frac{b-c}{b+c}N_0$$

Note, too, that, from the physical meaning of the problem, we must have $v''(x) \in C(-R, R)$, where R is an arbitrarily large number. Here $C(-R, R)$ is a space of functions continuous whenever $|x| \leqslant R$.

In the special case of a die at rest ($W = 0$), the problem corresponds to flexure of a plate on a Fuss–Winkler foundation, since when $W = 0$, the layer of a heavy ideal liquid behaves as a Fuss–Winkler foundation with the coefficient $k = \rho g$. In /2/, a closed solution of the flexure problem for a plate on a Fuss–Winkler foundation in the case $T = 0$ was obtained using a section partitioning method. The results of this paper lead us to conclude that contact forces arising between a die and the surface of the plate will be composed of a distributed load as well as concentrated forces acting on the edges of the line of contact. It has been proved /3/ that such a structure for contact forces is preserved even when $T \neq 0$.

2. Let us study the effect of die on a hydraulic foundation operating through a plate using the integral equations (1.9) and (1.10) obtained above. For this purpose, we will investigate the properties of its kernels and the structure of the solution. In view of the asymptotic behavior of $L(u)$,

$$L(u) = A + O(u^2)\ (u \to 0),\ A = (A_4 - A_3\beta^{-1})^{-1} \neq 0$$
$$L(u) = u^{-4} + O(u^{-6}) \quad (|u| \to \infty)$$

we may formulate the following lemma.

Lemma. For all values of $|z| \leqslant R$ (R is any arbitrarily large number), the representation

$$K(z) = \frac{\pi}{12}|z|^3 + F(z), \quad F(z) \in B_4{}^1(-R, R) \tag{2.1}$$

$$F(z) = \int_0^\infty \Big[L(u) - 2\sum_{k=1}^{2}\frac{a_kc_k^5}{u^2 - c_k^2}\Big]\Big(\cos uz - 1 - \frac{u^2z^2}{2}\Big)\frac{du}{u^4} +$$
$$\sum_{k=0}^{1}(-1)^k z^{2k}\Big[b_k - \frac{\pi z}{(2k+1)!}(a_1c_1^{2k+1} - a_2c_2^{2k+1})\Big] -$$
$$\pi\sum_{k=2}^{\infty}\frac{(-1)^k}{(2k+1)!}[a_1c_1^{2k+1}(|z|^{2k+1} + z^{2k+1}) + a_2c_2^{2k+1}(|z|^{2k+1} - z^{2k+1})]$$
$$b_0 = \int_0^\infty \Big[L(u) - 2\sum_{k=1}^{2}\frac{a_kc_k}{u^2 - c_k^2}\Big]du$$
$$b_1 = \int_0^\infty \Big[u^2L(u) - 2\sum_{k=1}^{2}\frac{a_kc_k^3}{u^2 - c_k^2}\Big]du$$

is valid.

Here $B_k{}^\mu(-R, R)$ is the space of fucntions whose k-th derivatives satisfy the Hölder condition on the segment $[-R, R]$ with index $0 < \mu \leqslant 1$, and the a_k are the residues of the function $L(u)$ at the poles c_k ($k = 1, 2$).

To prove the lemma, we will pass over to integration over the real axis in (1.10). We will have

$$K(z) = \int_0^\infty \Big[L(u) - 2\sum_{k=1}^{2}\frac{a_kc_k}{u^2 - c_k^2}\Big]\cos uz\,du - \pi\sum_{k=1}^{2}a_k[\sin c_k|z| - (-1)^k\sin c_kz]$$

Further, using the values of the integrals

$$\int_0^\infty (1 - \cos uz)\frac{du}{u^2} = \frac{\pi}{2}|z|, \quad \int_0^\infty \Big(\cos uz - 1 + \frac{u^2z^2}{2}\Big)\frac{du}{u^4} = \frac{\pi}{12}|z|^3$$
$$\int_0^\infty \Big(1 - \cos uz - \frac{u^2z^2}{2} + \frac{u^4z^4}{24}\Big)\frac{du}{u^6} = \frac{\pi}{240}|z|^5$$

we obtain (2.1).

Let us study the structure of the solution of this integral equation (1.9), (1.10). For this purpose, we consider the auxiliary equation

$$\int_{-1}^{1} |x-\xi|^3 g(\xi)\, d\xi = 12\psi(x) \quad (|x| \leqslant 1) \tag{2.2}$$

By the remarks in Sect.1, the solution $g(x)$ of the integral equation (2.2) must comprise as its terms delta functions at the points $x = \pm 1$, which would reflect the existence of concentrated forces in the contact forces at the edges of the line of contact. In addition, from the physical meaning of our problem, $v''(x) \in C(-R, R)$, as we have already noted. This condition imposes a constraint on the order of the generalized function $g(x)$. In light of the foregoing, we may state the following:

Theorem 1. If $\psi^{(4)}(x) \in C(-1, 1)$ in (2.2) and if the relations

$$\begin{gathered} 2\psi''(1) - 4\psi'(1) + 2\psi'(-1) + 3\psi(1) + 3\psi(-1) = 0 \\ 2\psi''(-1) + 4\psi'(-1) - 2\psi'(1) + 3\psi(1) + 3\psi(-1) = 0 \end{gathered} \tag{2.3}$$

are satisfied, then the solution of the integral equation (2.2) in a space of slowly increasing generalized functions Φ /3/ exists, is unique, and has the form

$$g(x) = \psi(x) + P_1\delta(x+1) + P_2\delta(x-1) \tag{2.4}$$

$$P_1 = \psi'''(-1) + {}^1/_2\,[\psi''(-1) + \psi''(1)], \quad P_2 = -\psi'''(1) + {}^1/_2\,[\psi''(1) + \psi''(-1)] \tag{2.5}$$

Note that the equations (2.3) which are conditions that state the solution of (2.2) (and, consequently, (1.9) and (1.10)) are bounded, may be used to define the unknown domain of contact of the die and the covering.

To prove the theorem, we verify that a function $g(x)$ of the form (2.4) satisfies equation (2.2). In fact, substituting it in (2.2) and using the well-known properties of the delta function /4/, we arrive at the conclusion that the integral equation (2.2) becomes an identity if (2.3) and (2.5) are satisfied. The uniqueness of our solution (2.4) follows from a well-known theorem (/4/, p.158).

Let us now rewrite the integral equation (1.9) in light of the representation (2.1) in the form

$$\int_{-1}^{1} g(\xi)\,|x-\xi|^3\, d\xi = 12\psi(x) \quad (|x| \leqslant 1) \tag{2.6}$$

$$\psi(x) = \lambda^3 \left[\delta + \alpha x - r(x) - \frac{1}{\pi}\int_{-1}^{1} g(\xi)\, F\left(\frac{x-\xi}{\lambda}\right) d\xi\right]$$

We assume that $r^{(4)}(x) \in C(-1, 1)$. If we now suppose that $g(x) \in \Phi$ (with order equal to zero), then by virtue of the properties of $F(z)$ given in the lemma, we will have $\psi^{(4)}(x) \in C(-1, 1)$. Hence, by Theorem 1, we conclude that Theorem 2 holds.

Theorem 2. Suppose $r^{(4)}(x) \in C(-1, 1)$ and let equations (2.3) hold. Then if the solution of the integral equation (2.6) in a space of slowly increasing generalized functions exists, it will have the form

$$g(x) = g^*(x) + P_1\delta(x+1) + P_2\delta(x-1) \tag{2.7}$$

while the function $g^*(x) \in C(-1, 1)$ also satisfies a Fredholm equation of the second kind with continuous kernel and continuous free term:

$$g^*(x) + \frac{1}{\pi\lambda}\int_{-1}^{1} g^*(\xi)\, F^{(1)}\left(\frac{x-\xi}{\lambda}\right) d\xi = -\lambda^3 r^{(4)}(x) - \frac{1}{\pi\lambda}\left[P_1 F^{(1)}\left(\frac{x+1}{\lambda}\right) + P_2 F^{(1)}\left(\frac{x-1}{\lambda}\right)\right] \quad (|x| \leqslant 1) \tag{2.8}$$

The constants $P_j\ (j = 1, 2)$ are given by (2.5), while the boundaries of the unknown region of contact between the die and the surface of the covering c and b are determined from (2.3).

To prove the theorem, assuming that the function $\psi(x)$ in (2.6) is known for the time being, we invert the integral operator in (2.6) with kernel $|x-\xi|^3$. By (2.4), we obtain an integral equation of the second kind for $g(x)$:

$$g(x)+\frac{1}{\pi\lambda}\int_{-1}^{1} g(\xi)F^{(4)}\left(\frac{x-\xi}{\lambda}\right)d\xi=-\lambda^3 r^{(4)}(x)+P_1\delta(x+1)+P_2\delta(x-1)\quad(|x|\leqslant 1) \tag{2.9}$$

The solution of this equation will be found in the form (2.7). Substituting (2.7) in (2.9) and using the properties of the delta function, we arrive at the integral equation (2.8). If equation (2.8) is solvable for a specified value $\lambda\in(0,\infty)$, the function $g^*(x)\in C(-1,1)$. The initial integral equation (2.6) is uniquely solvable in this case.

Theorem 3. In the class of functions $g^*(x)\in C(-1,1)\cap V(-1,1)$, the homogeneous equation (2.8)

$$g^*(x)+\frac{1}{\pi\lambda}\int_{-1}^{1} g^*(\xi)F^{(4)}\left(\frac{x-\xi}{\lambda}\right)d\xi=0\quad(|x|\leqslant 1) \tag{2.10}$$

does not have positive eigenvalues. Here $V(-1,1)$ is the space of functions with finite variation on the segment $[-1,1]$.

To prove the theorem, we introduce the Fourier transform of the function $g^*(x)$:

$$G^*(u)=\int_{-1}^{1} g^*(x)e^{iux}\,dx \tag{2.11}$$

and rewrite the homogeneous equation (2.10) as follows:

$$g^*(x)-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{[A_2\lambda^2u^2-A_3u\lambda\,\mathrm{cth}(\beta\lambda u)+A_4]L(\lambda u)+\right. \tag{2.12}$$
$$\left.2\sum_{k=1}^{2}\frac{a_kc_k^5}{\lambda^2u^2-c_k^2}\right\}G^*(u)e^{-iux}\,du-$$
$$\frac{1}{\lambda}\sum_{k=1}^{2}a_kc_k^4\int_{-1}^{1}g^*(\xi)\left[\sin\frac{c_k}{\lambda}|x-\xi|-(-1)^k\sin\frac{c_k}{\lambda}(x-\xi)\right]d\xi=0$$

Because of the properties of $g^*(x)$ given in the conditions of the theorem, the function $G^*(u)$ is at least continuous and has the estimate /5/

$$G^*(u)=O(|u|^{-1})\quad(|u|\to\infty) \tag{2.13}$$

Assuming that the nontrivial solution $g^*(x)$ exists for the homogeneous equation (2.12), we multiply (2.12) scalarly by $g^*(x)$. If $x-\xi>0$, in light of the Parceval equation /4/, we arrive at a relation of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lambda^4u^4L(\lambda u)-2\sum_{k=1}^{2}\frac{a_kc_k^5}{\lambda^2u^2-c_k^2}\right]|G^*(u)|^2\,du+ia_1c_1^4(|G^*(-c_1)|^2-|G^*(c_1)|^2)=0 \tag{2.14}$$

If $x-\xi<0$, a_1 in (2.14) must be replaced by a_2, and c_1 by c_2, and conversely.

Separating the real and imaginary parts in (2.14) and bearing in mind the fact that the integrand is sing-constant (this may be verified numerically for different parameters of the problem), we conclude that $G^*(u)\equiv 0$,, whence $g^*(x)\equiv 0$,, and the theorem is proved.

Now suppose that a die of thickness $2a$ has the corner points

$$x^{\mp}=(\mp 2a+c-b)(c+b)^{-1}$$

(further, for the sake of definiteness, we will consider the case of a die with a plane rectilinear base). Then, bearing in mind in the computations the fact that $v''(x)\in C(-R,R)$ as well as the arguments presented in /2/, we may conclude that the die cannot be brought into contact with the surface of the covering (1.3) in the neighborhoods of the angular points. Different variants in which the die is lifted off the plate may be found, in each case the form depending upon the actual mechanical characteristics of the problems under consideration, which we will review below.

Note that because of the foregoing and the results of Theorem 2, the solution of the integral equation (1.9) has in the general case the structure

$$g(x)=h(x)+P_3\delta(x-x^-)+P_4\delta(x-x^+) \tag{2.15}$$

and that the function $h(x)$ satisfies equation (1.9), where

$$r(x)=\frac{1}{\pi}\left[P_3K\left(\frac{x-x^-}{\lambda}\right)+P_4K\left(\frac{x-x^+}{\lambda}\right)\right]$$

and the values of the concentrated forces $P_j\ (j = 1,2)$ acting at the points $x = x^{\mp}\ (x^- < -1,\ x^+ > 1)$ are found from the linear algebraic system

$$K(0)P_3 + \int_{-1}^{1} h(\xi) K\left(\frac{x^- - \xi}{\lambda}\right) d\xi + K\left(\frac{x^- - x^+}{\lambda}\right) P_4 = \pi(\delta + \alpha x^-) \tag{2.16}$$

$$K(0)P_4 + \int_{-1}^{1} h(\xi) K\left(\frac{x^+ - \xi}{\lambda}\right) d\xi + K\left(\frac{x^+ - x^-}{\lambda}\right) P_3 = \pi(\delta + \alpha x^-)$$

3. Let us discuss the solution to the integral equation (1.9). For this purpose, we construct a rational function $L^*(\zeta)$ of the form

$$L^*(\zeta) = \frac{1}{\sigma_1(\zeta)\sigma_2(\zeta)} \prod_{n=1}^{N} \frac{\zeta^2 + z_n^2}{\zeta^2 + \zeta_n^2} \qquad (\sigma_j(\zeta) = \zeta^2 - c_j^2) \tag{3.1}$$

from the zeros and poles of the function $L(\zeta)$ which is meromorphic in the complex plane $\zeta = u + iv$. Here iz_n and $i\zeta_n$ are the zeros and poles of $L(\zeta)$, respectively, lying on the imaginary axis and having the asymptote

$$\zeta_n = z_n + O(n^{-3}) \qquad (n \to \infty) \tag{3.2}$$

(Here we are presenting results for the case in which $L(\zeta)$ has four single poles on the real axis).

Substituting (3.1) in (1.10), we obtain

$$K(z) = -2\pi\left[r_j \sin c_j |z| + \frac{1}{2}\sum_{n=1}^{N} R_n \exp(-\zeta_n |z|)\right] \tag{3.3}$$

$$j = 1\ (z > 0),\ j = 2\ (z < 0),\ r_j = \operatorname{Res}[L(\zeta), c_j]$$

$$R_n = i \operatorname{Res}[L^*(\zeta), i\zeta_n] \quad (n = 1, 2, \ldots, N)$$

Then the solution of equation (1.9), (3.3) will have the form

$$g(x) = T(x) + \sum_{n=1}^{N}\left[T_n^+ \exp\left(z_n \frac{x}{\lambda}\right) + T_n^- \exp\left(-z_n \frac{x}{\lambda}\right)\right] + P_1\delta(x+1) + P_2\delta(x-1) \tag{3.4}$$

in accordance with Theorem 2 and previous results /6/, i.e., (3.4) is the sum of a special solution of the nonhomogeneous differential equation

$$\prod_{n=1}^{N}\left(\frac{d^2}{dx^2} + z_n^2\right) T(x) = \left(\frac{d^2}{dx^2} - c_1^2\right)\left(\frac{d^2}{dx^2} - c_2^2\right) \prod_{n=1}^{N}\left(\frac{d^2}{dx^2} + \zeta_n^2\right)[\delta + \alpha x - r(x)] \tag{3.5}$$

and the general solution of the homogeneous equation (3.5). The values of the constants P_j $(j = 1,2)$ may be found from a system of the type (2.3), while the constants $T_n^{\pm}$ satisfy a linear algebraic system of equations of order $2N$, which may be obtained by subsituting (3.4) in (1.9) and (3.3) and by equating the coefficients of equal exponents in the resulting expression.

In the case of a parabolic die, for example, we obtain

$$r(x) = \gamma x^2, \quad T(x) = \sum_{i=0}^{2} b_i x^i, \quad b_2 = -\lambda\gamma[L^*(0)]^{-1}$$

$$b_0 = b_2[\lambda^2(c_1^{-2} - c_2^{-2}) - 1] - \frac{1}{2}\sum_{m=1}^{N}[T_m^+(\xi_{m1}^- + \xi_{m2}^+) + T_m^-(\xi_{m1}^+ + \xi_{m2}^-)]$$

$$b_1 = b_2\lambda^2(c_2^{-2} - c_1^{-2}) + \frac{1}{2}\sum_{m=1}^{N}[T_m^+(\xi_{m1}^- - \xi_{m2}^+) + T_m^-(\xi_{m1}^+ - \xi_{m2}^-)]$$

$$\xi_{mk}^{\pm} = c_k^2(c_k^2 + z_m^2)^{-1}\exp(\pm z_m/\lambda) \quad (k = 1, 2)$$

The system for determining the constants $T_n^{\pm}$ after the substitution

$$t_{2m} = T_m^+\left[z_m\left(1 + \frac{\lambda}{2\zeta_m}\right) - \frac{\lambda}{2}\right][(c_2^2 + z_m^2)(\zeta_m - z_m)]^{-1}\exp\left(\frac{z_m}{\lambda}\right)$$

$$t_{2m-1} = T_m^-\left[z_m\left(1 + \frac{\lambda}{2\zeta_m}\right) - \frac{\lambda}{2}\right][(c_1^2 + z_m^2)(\zeta_m - z_m)]^{-1}\exp\left(\frac{z_m}{\lambda}\right)$$

will have the form

$$t_j + \sum_{n=1}^{2N} a_{jn} t_n = d_j \tag{3.6}$$
$$d_{2j-1} = \lambda b_2 [2(1+\lambda\zeta_j^{-1}) + \lambda^2 (c_1^{-2} - c_2^{-2})] c_1^{-2} \zeta_j^{-1}$$
$$d_{2j} = -\lambda b_2 [2(1+\lambda\zeta_j^{-1}) + \lambda^2 (c_2^{-2} - c_1^{-2})] c_2^{-2} \zeta_j^{-1}$$

while for the coefficients a_{jn} we obtain for large enough j and n the asymptote (cf. (3.2))

$$a_{2j,\,2j},\, a_{2j-1,\,2j-1} = O(j^{-1} \exp(-2\pi\mu j)) \quad (\mu = aH^{-1}) \tag{3.7}$$
$$a_{2j,\,2n},\, a_{2j-1,\,2n-1} = O(j^{-1} n^{-3}); \quad a_{2j-1,\,2n},\, a_{2j,\,2n-1} = O(j^{-1} n^{-3})$$

Then, by (3.7) we may verify that the solution of the system (3.6) for large enough N will be close to the solution of the corresponding infinite system ($N = \infty$) relative to the norm of the space l_1 /8/.

Analogous results may be obtained for a plane die (because of their cumbersomeness, they will not be presented here). We only wish to note that in this case we must set $b_2 = 0$, and the expressions for the right sides will depend on the values of P_3 and P_4.

4. As an example, let us consider the motion problem (constant speed) for a die moving along the surface of a layer of ice. The ice will be assumed to be elastic, since the effect of creep in this case cannot have time to manifest itself and may be neglected. In the calculations, we will use the following physical /9/ and mechanical constants: $G = 2.4 \cdot 10^9$ N/m^2, $\gamma^* = 880$ kg/m^3, $\nu = 0.34$, $\rho = 1000$ kg/m^3, $\sigma = -5 \cdot 10^6$ N/m^2, $\epsilon = 0$, and $P = 10^3$ N/m.

In studying the motion of a plane die ($r(x) \equiv 0$) with values of the mechanical parameters under which the function $L(\zeta)$ does not have any poles on the real axis (for example, $W < W_c$ for a fixed value of the thickness of the plate, Fig.1), we find that at values $\mu < 0.1$ ($\beta = 3 \cdot 10^2$) the die will come into contact with the plate only at the two extreme points and that the maximal deflection of the plate beneath the die will occur at the middle. When $0.1 < \mu < 0.15$, the maximal deflections will be shifted from the center of the die, and the distance of the plate from the die will at the same time decrease. A further increase in the parameter μ will lead to contact between the die and the plate, and with $W < W_c$ (precritical mode), the zone of contact will be symmetric.

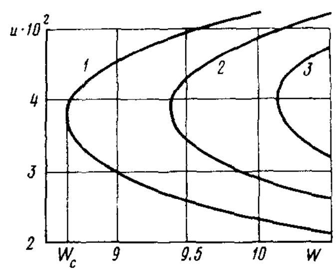

Fig.1

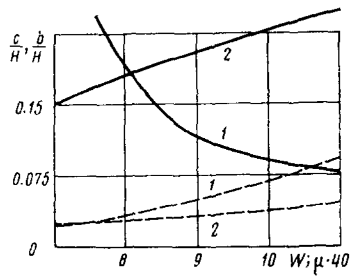

Fig.2

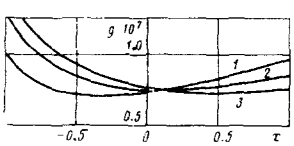

Fig.3

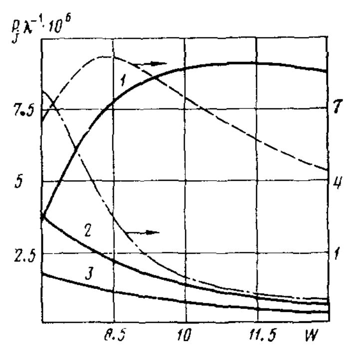

Fig.4

There is a more complicated contact mode if the speed of the die exceeds W_c. Then the function $L(\zeta)$ will have four single poles on the real axis, symmetrically located relative to the origin. In Fig.1 may be found curves depicting the variation in the positive poles c_1 and c_2 over some interval of variation of the speed (curves 1, 2, and 3 for $h = 0.25$; 0.30; and 0.35 m, respectively). If $L(\zeta)$ has a double pole on the real axis, this will correspond to W_c. In this case, the deflections under the die will increase without bound over time ($v(x) \sim \sqrt{t}$) /9, 10/. If $W > W_c$, the upper branches of the curves will correspond to the variation in c_2 pole, and the lower to $-c_1$. Note that the values of the real poles will essentially determine the wave pattern of the surface of the plate behind and in back of the die. Thus, the wavelength in front of the die will decrease with increasing speed, and increase behind the die. The die will be in contact with the plate over some interval $[-c, b]$ and touch it at its rear edge.

Curves depicting the variation of the zone of contact as a function of the speed of the die (solid curves) and as a function of the parameter μ (broken curves) are presented in Fig. 2 (curve 1 correspond to variations in the point c, and curve 2, to variations in point b). Fig.3 shows the variation in the contact pressure function beneath the die. It is clear that the contact pressure at the leading edge grows with increasing speed (curves 1,2 and 3 were obtained for $\mu = 0.4$ at speeds of 8.15, 8.65, and 9.15 m/s, respectively). The variation in the concentrated forces appearing at the edges of the line of contact are presented (solid curves) in Fig.4 (curves 1, 2, and 3 determine the forces at the points c, b, and a, respectively). The broken curve in Fig.4 shows the variation in the turning angle $\alpha\cdot 5\cdot 10^3$(rad)(the maximal value of the turning angle of the die corresponds approximately to the maximal forces at the leading edge of the die), while the dot-and-dash curve corresponds to the deflection of the plate beneath the die $\delta\cdot 10^4$ (m) (as the speed W approaches W_c, the deflections will grow markedly).

At a speed of the die $W_2 = \sqrt{gH}$, the function $L(\zeta)$ will have a double pole at the origin, which will indicate that deformation of the surface of the plate (understood as a rigid unit) will be observed together with a wave pattern on the surface. When $W > W_2$, the rear edge of the die will separate from the plate and contact will be present only along the segment $[-c, b]$. As the speed of the die increases over the interval $W_2 < W < W^*$ ($\beta = 2\cdot 10^2$, $W^* =$ 20.6 m/s), the region of contact will increase, and when $W > W^*$ start to decrease. The turning angles of the die will decrease with increasing speed of the die. Note that a change in the contact modes between the die and the plate will be related to qualitative variations in the function $L(\zeta)$.

The authors wish to express their appreciation to V.M. Aleksandrov for his interest in the present study.

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Translated by R.H.S.